Math 31 - Homework 6

Due Friday, August 10

Easy

1. Classify all abelian groups of order 600 up to isomorphism.

2. Prove that if G_1 and G_2 are abelian groups, then $G_1 \times G_2$ is abelian.

3. [Herstein, Section 2.9 #1] If G_1 and G_2 are groups, prove that $G_1 \times G_2 \cong G_2 \times G_1$.

4. [Herstein, Section 4.2 #2] Let R be an integral domain. If $a, b, c \in R$ with $a \neq 0$ and $ab = ac$, show that $b = c$.

5. [Herstein, Section 4.1 #13] Find the following products of quaternions.

(a) $(i + j)(i - j)$.

(b) $(1 - i + 2j - 2k)(1 + 2i - 4j + 6k)$.

(c) $(2i - 3j + 4k)^2$.

(d) $i(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k) - (\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)i$.

Medium

6. Let G_1 and G_2 be finite groups.

(a) If $a_1 \in G_1$ and $a_2 \in G_2$, prove that

$$|(a_1, a_2)| = \text{lcm}(|a_1|, |a_2|),$$

where $\text{lcm}(m, n)$ denotes the least common multiple of the integers m and n.

(b) [Herstein, Section 2.9 #2] Prove that if G_1 is a cyclic group of order n and G_2 is a cyclic group of order m, then $G_1 \times G_2$ is a cyclic group if and only if $\gcd(m, n) = 1$. [**Hint:** $mn = \text{lcm}(mn) \cdot \gcd(m, n)$.] (Note that this implies the theorem that was mentioned in class: if m and n are relatively prime, then $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$.)

7. [Herstein, Section 4.1 #4] Recall that if $a \in \mathbb{Q}$, we can always write a in **lowest terms** (or **reduced form**) as $a = m/n$, where $\gcd(m, n) = 1$. Define

$$R = \left\{ \frac{m}{n} \in \mathbb{Q} : \gcd(m, n) = 1 \text{ and } n \text{ is odd} \right\}.$$

That is, R is the set of all rationals which, when written in lowest terms, have an odd denominator. Show that R is a ring under the usual addition and multiplication of rational numbers. Determine which elements of R are units. (This ring R is actually quite important in higher algebra. It is usually denoted by $\mathbb{Z}_{(2)}$, and called the *localization* of $\mathbb{Z}$ at 2.)

8. [Herstein, Section 4.1 #21] Show that any field is an integral domain.

9. [Herstein, Section 4.2 #3] Let R be a finite integral domain with identity $1 \in R$. Show that R is actually a field.

Hard

10. Let D_n denote the dihedral group of order $2n$, let $r \in D_n$ denote the counterclockwise rotation by $2\pi/n$ radians, and let m denote any reflection of the regular n-gon. Recall that the rotation subgroup

$$H = \{e, r, r^2, \ldots, r^{n-1}\}$$

is a normal subgroup of D_n. Let $K = \{e, m\}$; we saw in class that K is a subgroup, but it is not normal. You will need two facts regarding D_n (which you do not need to prove):

1. Every element of D_n can be written as $r^i m^j$, with $0 \le i \le n-1$ and $j = 0$ or 1. Thus $D_n = HK$.
2. You proved earlier (in a special case) that $mr = r^{-1}m$.

Define a group G as follows: the elements of G are pairs (r^i, m^j) (so that $G = H \times K$ *as sets*), but the binary operation on G is "twisted" in some sense. More specifically, we define

$$\begin{aligned}(r^i, e)(r^j, e) &= (r^{i+j}, e)\\ (r^i, m)(r^j, e) &= (r^{i-j}, m)\\ (r^i, e)(r^j, m) &= (r^{i+j}, m)\\ (r^i, m)(r^j, m) &= (r^{i-j}, e)\end{aligned}$$

for all i, j between 1 and $n-1$. Now define $\varphi : G \to D_n$ by

$$\varphi(r^i, m^j) = r^i m^j.$$

Prove that φ is an isomorphism of G onto D_n. (This shows that D_n is not quite a direct product of the subgroups H and K, since K isn't normal. However, things can be made to work if we modify the multiplication on $H \times K$ slightly.)